Euler characteristic of CW-complex

1. Let X be a finite CW-complex and $\{C_p(X), \partial\}$ be its cellular chain complex of finitely generated free R-modules. (R : PID)

We know

$$\begin{aligned}\chi(X) &= \sum(-1)^p \mathrm{rk} H_p(X)\\ &= \sum(-1)^p \mathrm{rk} C_p(X)\\ &= \sum(-1)^p \sharp\{p-\text{cells in } X\}\end{aligned}$$

This formula is more convenient in practice.

Example.
(1) S^2: one 0-cell, one 2-cell $\Rightarrow \chi(S^2) = 1+1 = 2$
or one 0-cell, one 1-cell, and two 2-cells $\Rightarrow \chi(S^2) = 1-1+2 = 2$
(2) T^2: one 0-cell, two 1-cells, and one 2-cell $\Rightarrow \chi(T^2) = 1-2+1 = 0$
(3) $\chi(L(p,q)) = 1-1+1-1 = 0$
(4) $\chi(\mathbb{R}P^n) = \begin{cases} 0 & n \text{ is odd} \\ 1 & n \text{ is even} \end{cases}$
(5) $\chi(\mathbb{C}P^n) = n+1$, $\chi(\mathbb{H}P^n) = n+1$

2. X, Y : finite CW-complex $\Rightarrow X \times Y$: CW-complex.
(In general, one of X, Y is locally compact.)
증명 $X = \{e_\alpha\}$, $Y = \{e_\beta\} \Rightarrow \{e_\alpha \times e_\beta\}$ is a cell decomposition of $X \times Y$.
Check the detail.(Exercise) □

정리 1 $\chi(X \times Y) = \chi(X) \times \chi(Y)$

증명 Let n_k be the number of k-cells in X, and m_l the number of l-cells in Y.

$$\begin{aligned}\chi(X \times Y) &= \textstyle\sum_p (-1)^p (\sum_k n_k m_{p-k})\\ &= \textstyle\sum_{k,l} (-1)^{k+l} n_k m_l\\ &= (\textstyle\sum_k (-1)^k n_k)(\sum_l (-1)^l m_l)\\ &= \chi(X) \times \chi(Y)\end{aligned}$$

□

e.g., $\chi(M \times S^1) = 0$

3. $Z = X \cup_f Y$, (X, A): a collared pair. $f : A \to Y$.
$\Rightarrow \chi(Z) = \chi(X) + \chi(Y) - \chi(A)$.
증명 Exact sequence

$$\cdots \longrightarrow H_q(A) \longrightarrow H_q(X) \bigoplus H_q(Y) \longrightarrow H_q(Z) \longrightarrow H_{q-1}(A) \longrightarrow \cdots$$

로부터 자명하다. □

일반적으로 exact sequence L

$$\cdots \longrightarrow A_i \longrightarrow B_i \longrightarrow C_i \longrightarrow A_{i-1} \longrightarrow \cdots$$

가 존재하면

$$0 = \chi(L) = \chi(C) - \chi(B) + \chi(A)$$

임을 확인할 수 있다.

Example. $\chi(M^n \sharp N^n)$
Let $M' := M - \{n\text{−ball}\}$. Then

$$\chi(M) = \chi(M') + 1 - \chi(S^{n-1})$$

Therefore

$$\begin{aligned} \chi(M^n \sharp N^n) &= \chi(M') + \chi(N') - \chi(S^{n-1}) \\ &= \{\chi(M) - 1 + \chi(S^{n-1})\} + \{\chi(N) - 1 + \chi(S^{n-1})\} - \chi(S^{n-1}) \\ &= \chi(M) + \chi(N) - 2 + \chi(S^{n-1}) \\ &= \begin{cases} \chi(M) + \chi(N), & n\text{: odd} \\ \chi(M) + \chi(N) - 2, & n\text{: even} \end{cases} \end{aligned}$$